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Sintesi

The Semilinear Klein-Gordon equation in two and three space dimensions

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In the thesis we will discuss some recent developments for a particular semilinear defocusing wave equation: the semilinear $u|u|^{p-1}$ Klein-Gordon equation, focusing on the two and three space dimensions cases.
A semilinear wave equation (SLW) has the form

$$\square u(t,x) + g(u(t,x)) = 0 \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{0.1}$$

where $g : \mathbb{R} \to \mathbb{R}$, the differential operator $\square$ is defined $\square := \partial_{tt} - \Delta$ with the standard space Laplace operator $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.
The Cauchy problem for the above equation on $\mathbb{R} \times \mathbb{R}^n$ has initial data:

$$(u, u_t)|_{t=0} = (\varphi, \psi) \tag{0.2}$$

We can associate to SLW, the energy:

$$E(u)(t) = \int_{\mathbb{R}^n} e(u)(t,x)dx \tag{0.3}$$

where $e(u)(t,x) = \left(\frac{|\nabla u|^2}{2} + \frac{|u_t|^2}{2} + G(u)\right)$,$G(u) = \int_0^u g(s)ds$ and $E_0 := E(u)(0)$ is the initial data energy. The semilinear Klein-Gordon equation is a SLW with odd nonlinearity $g(u) = u|u|^{p-1}$.
In the last 25 years there has been considerable interest in studying hyperbolic non linear partial differential equations. They model phenomena of wave propagation in different problems arising from Physics: for instance, water waves, lasers, problems of general relativity and relativistic particle Physics.
In 1920's Physicists were looking for relativistic quantum mechanic equation for the electron. The Klein-Gordon equation was proposed by Klein and Gordon in 1927 but it did not work. Together with a positive energy solution, by the relativistic invariant

$$E^2 = m^2c^4 + p^2c^2,$$

a negative energy solution can be found and it can not be a priori discarded. The next year the equation proposed by Dirac was able to describe the spinning electron behaviour.

In 1934 the Klein-Gordon equation was re-considered together with the study of spinless particles. It was discovered that it describes the behaviour of the lightest mesons, the π-mesons particles. In the free form (without potential) the Klein-Gordon equation assumes the form:

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = \frac{m^2c^2}{\hbar^2}\psi$$

where c is the speed of light, $\hbar$ is the Plank constant divided by 2π, m is the mass of the particle. By the Pauli and Weisskopf interpretation, the Klein-Gordon equation is a field equation for a charged spin-0 field. It is interesting how this equation put together the relativistic constant c and the quantum mechanic one $\hbar$.

Assuming $c = 1$, the equation can be rewritten in the following way:

$$\left(\Box + \frac{m^2}{\hbar^2}\right)\phi = 0$$

the box operator$\Box$, or d'Alambertian operator, emphasizes that there are no more differences between time and space variables with the Minkowski metric. The Minkowski metric is defined by the quadratic form

$$\eta = \begin{pmatrix} -1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ & & ... & \\ 0 & 0 & ... & 1 \end{pmatrix} = \text{diag}(-1, 1, ..., 1) \in \mathcal{M}_{m+1,m+1}(\mathbb{R})$$

for the variables$(t, x_0, ..., x_n) = (y_0, ...y_n) \in \mathbb{R}^{n+1}$.

Under a more analytic point of view it is important to interpret the solution of a wave equation in the following sense:

Definition 0.1. Let $\mathbf{X}$ be a Banach space and $u(t, x)$ be a solution for the Cauchy problem (0.1)+(0.2) with $(\varphi, \psi) \in \mathbf{X}$.

We can define

$$U(t): \begin{array}{ccc} [0,T] & \longrightarrow & \mathbf{X} \\ t & \longmapsto & (u(t,\cdot), u_t(t,\cdot)) \end{array}$$

Henceforth, for the sake of notational simplicity, we will identify $U(t)$ with $u(t)$.
We will concentrate on *defocusing* semilinear wave equations. We say that a semilinear wave equation is *defocusing* if $G(u)$ cooperates with the energy of the linear operator, hence if the whole energy is positive defined.
For example, consider the defocusing semi linear Klein-Gordon problem with power nonlinearity:

$$\Box u + u|u|^{p-1} = 0 \qquad \mathbb{R} \times \mathbb{R}^n. \tag{0.4}$$

with initial data (0.2). The associated energy is

$$E(u)(t) = \int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} + \frac{u_t^2}{2} + \frac{|u|^{p+1}}{p+1} \right) dx \geq 0.$$

Generally speaking, a defocusing wave equation tends to dissipate the solution when it is concentrated. It follows that defocusing equations will tend to avoid blow up phenomena. We say that a semilinear wave equation is *focusing* if the energy associated is not positive defined, for example if $G(u)$, signed defined, has the wrong sign with respect to the linear operator energy. This type of equations tend to be affected by blow-up phenomena. Changing sign to the nonlinearity we have the focusing problem:

$$\Box u - u|u|^{p-1} = 0 \qquad \mathbb{R} \times \mathbb{R}^n. \tag{0.5}$$

with initial data (0.2) and associated energy $E(u)(t) = \int_{\mathbb{R}^n} (\frac{|\nabla u|^2}{2} + \frac{u_t^2}{2} - \frac{|u|^{p+1}}{p+1}) dx$. Note that in the latter case the primitive of the nonlinearity in the energy expression has a minus sign.
It is important to focus on the main aspects and on the different issues of the SLW studies. Consider the following classical definitions of well and ill posed semilinear problems:

Definition 0.2. Consider the Cauchy problem (0.1)+(0.2).

Let $\mathbf{X}$ be a Banach space.

- We say that the Cauchy problem, with $(\varphi, \psi) \in \mathbf{X}$ is *locally-well posed* in $\mathbf{X}$ if there exists $T > 0$, a space $\mathbf{Y}(T)$ and a $u \in \mathbf{Y}(T)$ such that:

  1. u satisfies the equation in the sense of distributions on $[0, T] \times \mathbb{R}^n$:
  $$\int_0^T \int_{\mathbb{R}^n} u \Box v dz = \int_0^T \int_{\mathbb{R}^n} gv + \int_{\mathbb{R}^n} \phi v_t(\cdot, 0) dx - \int_{\mathbb{R}^n} \psi v(\cdot, 0) dx \quad (0.6)$$
  $\forall v \in C_0^\infty((-\infty, T] \times \mathbb{R}^n)$
  2. u is unique in $\mathbf{Y}(T)$
  3. For any $(\varphi, \psi) \in \mathbf{X}$ there is a neighbourhood N of $(\varphi, \psi) \in \mathbf{X}$ such that the solution map
  $$(\overline{\varphi}, \overline{\psi}) \longmapsto \overline{u}$$
  is a continuous function from N to $\mathbf{Y}(T)$.

- We will say that the Cauchy problem is *globally well-posed* in $\mathbf{X}$ if it is well posed and T can be taken arbitrary.

- We will say that the Cauchy problem is *strongly well-posed* in $\mathbf{X}$ if it is well posed and the solution map is uniformly continuous

- We will say that the Cauchy problem is *ill-posed* in $\mathbf{X}$ if the solution map is not continuous.

- We will say that the Cauchy problem is *weakly ill-posed on a set* $\mathbf{Z} \subset \mathbf{X}$ if the solution map is not uniformly continuous on $\mathbf{Z}$.

We will consider $\mathbf{Y}(T)$ always embedded $C([0, T], \mathbf{X})$. We will see that the natural Banach space for the well-posedness for $n \geq 3$ will be the energy space $\dot{H}^1 \times L^2$

defined by the norm $\|u(t)\|_{\dot{H}^1\times L^2} = \|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}$ and for $n = 2$ is the space $H^1 \times L^2$ defined by $\|u(t)\|_{\dot{H}^1\times L^2} = \|u(t)\|_{H^1} + \|u_t(t)\|_{L^2}$.

Pioneers in studying the well posedness of wave equations were Strichartz [38], Peacher [29], Ginibre and Velo [11]. In the late 1980's there were introduced tools from harmonic analysis to study the local theory of Cauchy problems (see [24],[23]). For more modern results see Tao [44], or Tataru [47],[46]. In the recent years there has been a great interest in studying dispersive wave equations, their long time behaviour, regularity and uniqueness of solutions. For the 3 space dimensions case, the results we will present are essentially due to Segal[31], Jörgens[21], Rauch[30], Strichartz[38],Struwe [39],[40]. In 2 space dimensions results of regularity and well posedness of semilinear Klein-Gordon equation are mainly provided by Struwe([43],[42],[41]) Ibrahim, Majdoub and Masmoudi ([16], [15], [18]). All these results have been provided in the last two decades, especially the 2 space dimensions problem has been developed in the last decade. We will see that although techniques are particularly developed there are completely open problems.

At this point, for $n \geq 3$, it is important to classify the semilinear Klein-Gordon equation introducing the *critical exponent*. The problem at the critical exponent heuristically comes from how the energy scales with respect to the scaling:

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x) \qquad p \in [1, +\infty).$$

Let u be a classical solution of (0.4):

$$\begin{aligned} \Box u_\lambda(t,x) &= \lambda^{\frac{2}{p-1}+2} \Box u(\lambda t, \lambda x) \\ &= \lambda^{\frac{2}{p-1}+2} u(\lambda x, \lambda t)|u(\lambda t, \lambda x)|^{p-1} \\ &= \lambda^{\frac{2}{p-1}+2-\frac{2}{p-1}-(p-1)(\frac{2}{p-1})} u_\lambda(t,x)|u_\lambda(t,x)|^{p-1} \\ &= u_\lambda(t,x)|u_\lambda(t,x)|^{p-1} \end{aligned}$$

hence both $u,\ u_\lambda$ solve (0.4). The associated energy:

$$E(u(t)) = \int_{\mathbb{R}^n} \left(\frac{|u_t|^2 + |\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1} \right) dx$$

evaluated on u_λ changes:

$$E(u_\lambda) = \lambda^{2-n+\frac{4}{p-1}} \int_{\mathbb{R}^n} \left(\frac{|u_t|^2 + |\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1} \right) dx = \lambda^{2-n+\frac{4}{p-1}} E(u).$$

We note that $E(u_\lambda) = E(u)$,i.e. the energy does not change under scaling, if $p = 2^* - 1$ where $2^* = \frac{2n}{n-2}$. Moreover the exponent $2 - n + \frac{4}{p-1}$ is positive for $p < 2^* - 1$ and is negative for $p > 2^* - 1$. It is clear that that for $\lambda \to +\infty$ for $p \geq 2^* - 1$ the energy remains bounded (actually goes to zero for $p > 2^* - 1$). It follows that the solution's energy concentrates at a point for $p < 2^* - 1$. The concentration implies an infinite amount of energy and in this case it might be possible to prove well posendess or regularity of solutions. For $n \geq 3$ and $p = 2^* - 1$ the concentration requires a finite amount of energy and it should be ruled out for small energy data. Ginibre and Velo [10] show that the problem (0.4)+(0.2) is globally well posed in the energy space in the sub-critical range $p < 2^* - 1$. For the critical power nonlinearity the problem was solved by Shatah and Struwe. In [33] they proved that the critical problem is well posed in the energy space for any space dimensions $n \geq 3$. In the super-critical range the well-posedness is an open problem except for some partial results(for example see [5],[6],[17],[26]).

In two space dimensions, power nonlinearities appear to be sub-critical with respect to H^1 norm. The critical exponent in Sobolev embedding is infinite. Hence the exponential nonlinearity seems to be a critical nonlinearity. The 2D Klein-Gordon equation we will study is the following:

$$\Box u + ue^{u^2} = 0 \tag{0.7}$$

with associated energy density $e(u) = \frac{1}{2}\left(u_t^2 + |\nabla u|^2 + e^{u^2} - 1\right)$.

The first theorems about this type of equation were due to Ibrahim Majdoub and

Masmoudi (see for example [16], [18],) and they make evidence of the relation between the considered equation and its initial energy data by the version of Moser Trudinger inequality provided by Adachi and Tanaka:

Proposition 0.3. Let $\alpha \in (0, 4\pi)$. It exists a constant $C = C(\alpha)$ such that

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2(x)} - 1\right) dx \leq C\|u\|_{L^2}^2 \tag{0.8}$$

for all $u \in H^1(\mathbb{R}^2)$ such that $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$.
If $\alpha \geq 4\pi$ the inequality is false.

Remark 0.4. $\alpha = 4\pi$ becomes admissible if we require $\|u\|_{H^1} \leq 1$. In particular

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1\right) dx < +\infty$$

and it is false for $\alpha > 4\pi$. See [1].

It seems that problems arises when we consider "large" initial data energy. In fact in 2006 Ibrahim Majdoub and Masmoudi [16] proved the following theorem:

Theorem 0.5. *Assume* $\|\nabla\varphi\|_{L^2} < 2\pi$.
It exists a time $T > 0$ *and a unique weak solution* $u(t, x)$ *to the problem (0.7) with initial data (0.2) in* $H^1 \times L^2$ *in*

$$C_T(H^1(\mathbb{R}^2)) \cap C_T^1(L^2(\mathbb{R}^2)).$$

Furthermore $u \in L_T^4(C^{\frac{1}{4}}(\mathbb{R}^2))$, $\forall t \in [0, T)$ *and*

$$E(u)(t) = E(u)(0)$$

The proof is essentially based on a contraction argument. If $E(u)(0) \leq 2\pi$ it is possible to prolong the solution for any $T > 0$, hence to have a global weak solution. The authors refers that the contraction argument is enough to assure global strong well-posedness in $H^1 \times L^2$ for "small" energy initial data , $E(u)(0) \leq$

2π. In this sense we define *critical* the Cauchy problem (0.7) + (0.2) with initial data energy 2π, *sub-critical* with initial data energy less than 2π and *super-critical* with initial data energy larger than 2π. Asking higher regularity to initial data it is possible to remove the smallness condition on the initial data and have a local well-posedness result for strong solutions in the space $H^s \times H^{s-1}$ with $s > 1$. Using the last result, in 2009, Ibrahim, Majdoub and Masmoudi showed that for any initial data the problem (0.7)+(0.2) is locally well posed in $H^1 \times L^2$(see[18]). In addition, in [17] and in [18], they proved, in the super-critical case, that the local solution does not depend on the initial data in a locally uniform continuous fashion, i.e. the problem is weakly ill posed on $\{E < 2\pi + \delta\} \subset H^1 \times L^2$, $\delta > 0$. In [18], they consider the re-scaled equation

$$\Box u + ue^{4\pi u^2} = 0$$

with associated energy

$$E(u)(t) = \int_{\mathbb{R}^2} \left(|u_t|^2 + |\nabla u|^2 + \frac{e^{4\pi u^2} - 1}{4\pi} \right) dx.$$

The critical energy become 1 and we have the following ill-posedness theorem in $\{E < 1 + \delta\} \subset H^1 \times L^2$:

Theorem 0.6. *Let $v > 0$. It exists a sequence of positive real numbers $\{t_k\}$ and two sequences$\{u_k\}_{k\in\mathbb{N}}$,$\{w_k\}_{k\in\mathbb{N}}$ solving the Klein-Gordon equation (0.7)such that*

$$t_k \overset{k\to+\infty}{\to} 0$$

$$\|(u_k - w_k)(0)\|^2_{H^1} + \|\partial_t(u_k - w_k)(0)\|^2_{L^2} \le o(1) \quad \text{as } k \to +\infty$$

$$0 < E(u_k)(0) - 1 \le e^3 v^2, \quad 0 < E(w_k)(0) - 1 \le v^2$$

and

$$\liminf_{k\to+\infty} \|\partial_t(u_k - w_k)(t_k)\|^2_{L^2} \ge \frac{\pi}{4}(e^2 + e^{3-8\pi})v^2$$

It follows that the problem is weakly ill posed in the energy space in the supercritical range.
In the thesis we will present the 2006 result in order to understand how the Moser Trudinger inequality implies the restriction on the initial data and we will provide the example of the ill posedness for large energy data.
Other important studies we survey in the thesis strictly concern the existence and regularity of solutions for SLW. The first part of the thesis is in fact devoted to classical and modern results for general SLW always directed to the study of Klein-Gordon equation.
At this point it is important to have a broad overview on some important SLW properties. We will present the Lagrangian formulation for the wave equation in order to understand the relation between invariants and conserved momenta.
The SLW can be obtained by Euler-Lagrange equations for the action

$$\mathcal{A} = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \left\{ -\frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + G(u) \right\} dx dt.$$

The Lagrangian formulation, of these wave equations, allows us to systematically derive conserved quantities by means of Noether's theorem

Theorem 0.7. *If the action $\mathcal{A}$ is invariant under a group of diffeomorphisms, the solution u satisfies conservation laws determinated by this invariance.*

Another important property of SLW is the finite speed of signal's spread. It means that if the problem's solution has initial data compactly supported in a subset $\Omega \Subset \mathbb{R}^n$, its support at the time t will be contained in the envelop $\Omega + B_t(0)$, for $t \geq 0$. In particular if $\Omega = B_r(0)$, then *Supp* $u \subset K^+(-r, 0) = \{(t, x) || x| \leq r + t, t \geq 0\}$, where $K^+(-r, 0)$ is the forward truncated cone with vertex in $(-r, 0)$. Moreover, considering the backward light cone $K(z_0) = \{(t, x)| \ |x - x_0| \leq t_0 - t, 0 \leq t \leq t_0\}$ we are sure that the perturbation outside the cone will not affect the solution inside. This will make us able to reduce global problems, for instance existence

of global solution, to a local estimate of solution's norm.
Segal, in [31], shows that it is possible to find global weak solutions for general SLW assuming initial data (0.2) in $(H^1 \times L^2)(\mathbb{R}^n)$, $G(\varphi) \in L^1(\mathbb{R}^n)$ and asking coerciveness on the nonlinearity:

Theorem 0.8 (Segal's Theorem). *Consider SLW . Assume $g(0) = 0$ and the coercive assumption on g:*

$$0 \le ug(u) + C_1u^2 \le CG(u) + C_2u^2 \qquad \text{for } |u| \to +\infty \tag{0.9}$$

with $C, C_1, C_2 \in [0, +\infty)$. For any initial data $(\varphi, \psi) \in (H^1 \times L^2)(\mathbb{R}^n)$, $G(\varphi) \in L^1(\mathbb{R}^n)$, it exists a global weak solution for SLW problem $u \in H^1_{loc}$ such that $Du \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$, $G(u) \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^n))$ and

$$E(u)(t) \le E(u)(0) = \int_{\mathbb{R}^n} \frac{|\psi(t,x)|^2}{2} + \frac{|\nabla\varphi(t,x)|^2}{2} + G(\varphi(t,x))dx =: E_0 \tag{0.10}$$

For the proof, we need to show the existence of a unique weak solution for Lipschitz nonlinearity in order to use approximated problem with globally Lipschitz nonlinearity.
The Segal's result provides the existence of global weak solutions for any $u|u|^{p-1}$ nonlinearity. On the other hand it does not work with the 2 space dimensions exponential nonlinearity considered.
The main problem seems to establish regularity and uniqueness conditions on solutions. Following Shatah and Struwe [35], by using energy methods we provide uniqueness and the regularity of this solutions for $1 \le p \le 1 + \frac{2}{n-2}$. Ginibre and Velo, in [10], were able to improve the uniqueness range for solutions for the problem (0.4)+(0.2) in the energy space $\dot{H}^1 \times L^2$ up to the Sobolev critical exponent $2^* - 1$. Regarding existence of smooth solution Shatah and Struwe, in [33], show that a unique smooth solution of the smooth Cauchy problem can be found until $n = 7$. Struwe, in [35], quoted that for $p \le 2^* - 1$ and $n \le 9$ for

smooth initial data we have a smooth solution.

Concerning the focusing case, global regularity and well posedness conjecture fails. In fact H.Levine[28], for instance, shows that if $(\varphi,\psi) \in \dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, the initial data energy is negative and we have a growth condition on nonlinearity, the solution has finite lifespan.

Theorem 0.9 (Levine's Theorem)**.** *Consider the initial value problem (0.1)+(0.2). Assume that*

$$ug(u) \leq (2+\epsilon)G(u) \qquad \forall u \in \mathbb{R},\ \exists \epsilon > 0 \tag{0.11}$$

Assume that the initial data satisfies $E(u)(0) = \int_{\mathbb{R}^n}\left(\frac{|\nabla\varphi|^2}{2} + \frac{\psi^2}{2} + G(\varphi)\right)dx < 0$. *Then the solution of the SWE does not exists for all the time.*

For focusing problem we have a ground state conjecture: there exist a "ground state", whose energy is a threshold for global existence. Merle and Kenig [25], [22] proved the theorem:

Theorem 0.10. *For the focusing energy critical SLW,* $3 \leq n \leq 5$, $(\varphi,\psi) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, $E(\varphi,\psi) < E(W,0)$, *where* $W(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-\frac{n-2}{2}} \in \dot{H}^1$ *we have:*

- *if* $||\varphi||_{\dot{H}^1} < ||W||_{\dot{H}^1}$, *the solution exists for all the time.*
- *if* $||\varphi||_{\dot{H}^1} > ||W||_{\dot{H}^1}$, *the solution has finite lifespan.*

As we said, in the thesis we focus on the three dimensional (0.4)+(0.2) problem. In this case the critical exponent is 5 and as above we can distinguish a *subcritical* ($p \leq 5$), a *critical* ($p = 5$) and a *supercritical* ($p \geq 5$) equation. Following Jörgens's idea, presented in [21], we will be able to show the theorem:

Theorem 0.11. *Let G be a closed bounded domain. Let* $\varphi \in C^3(G), \psi \in C^2(G)$ *and* $g \in C^2(\mathbb{R})$, $g(0) = 0$. *Then it exists* $T > 0$ *and a unique* $u \in C^2(K^T(G))$ *solution of the problem (0.1)+(0.2) in* $K^T(G)$.

where $K^T(G) = (\cup_{B\subset G} K(B)) \cap ([0,T]\times \mathbb{R}^3)$, B are $\mathbb{R}^3$ balls and $K(B)$ is the backward cone built on the ball B.
It gives the existence of a local classical solution for any regular initial data and nonlinearity. We give a proof based on a converging argument of approximate solutions. Then, following [35], we will prove the existence of a unique global classical solution in the subcritical range. In particular we can observe that blow up phenomena are characterized by the loss of L^∞ norm of solution. This important result can not be generalized in higher dimensions because it heavily uses representation formula for solutions of the linear Cauchy problem.
Completed the subcritical case, we consider the 3D-SLW with critical nonlinearity. Assuming small energy initial data, Rauch in [30], was able to show the existence of an L^∞ a priori bound on solutions depending only on the initial data and its support's size. It prevents the overcome of blow up at finite time and ensures the existence of a unique classical solution:

Theorem 0.12. *It exists $\epsilon_0 > 0$ such that the initial data problem (0.4)+(0.2) with $p = 5$, $\varphi \in C^3(\mathbb{R}^3)$, $\psi \in C^2(\mathbb{R}^3)$ such that initial data energy $E(u)(0) \leq \epsilon_0$ admits a unique C^2 global solution.*

The bound obtained comes from the explicit representation formula for solution of the linear three space dimensions problem. On every backward cone $K(z_0)$ such that $u(z_0) = \sup_{K(z_0)} |u(t,x)|$ we have the estimate:

$$|u(z_0)| \leq C + \frac{u(z_0)}{4\pi}\int_{M_T(z_0)} \frac{u^4(t,x)}{t_0 - t} d\sigma + CE_0^{\frac{5}{6}} |t_0 - T|^{-\frac{1}{2}} \quad 0 < T < t_0 \tag{0.12}$$

where $M_T(z_0)$ is the mantle of the cone's upper part higher than T.
For small energy initial data we can choose $T = 0$ and have $\frac{1}{4\pi}\int_{M_0(z_0)} \frac{u^4(t,x)}{t_0-t} d\sigma < 1$. This gives a uniform control on the L^∞ norm of the solution over every backward light cone and assure the absence of blow up. Afterwards, following Struwe

[40] we will see the existence of a unique global classical solution for any initial data by the Grillakis' geometric idea presented in [13], assuring the condition:

$$\limsup_{z\to z_0, z\in K(z_0)} \int_{M_T(z)} \frac{u^4}{t_0 - t} d\sigma \leq 2\pi.$$

We can prevent the blow up in z_0 through the above condition and (0.12): it suffices to have a uniform bound for u on $K(z_0)\backslash\{z_0\}$ i.e. the condition

$$\limsup_{z\to z_0, z\in K(z_0)} |u(z)| < +\infty.$$

Struwe's theorem reads as follows:

Theorem 0.13. *Let $\varphi \in C^3(\mathbb{R}^3), \psi \in C^2(\mathbb{R}^3)$. It exists a unique, global, classical solution, i.e. $u \in C^2([0, +\infty) \times \mathbb{R}^3)$, of the Cauchy problem:*

$$\begin{cases} u_{tt} - \Delta u + u^5 = 0 \\ u|_{t=0} = \varphi \\ u_t|_{t=0} = \psi \end{cases} \tag{0.13}$$

Note that the bound that will be obtained is not an a priori bound for the solution u on a general cone K only depending on initial data and K.
Eventually we will present the pointwise decay property of the solution of the u^5 Klein-Gordon equation's as in Grillakis' [13].

Theorem 0.14. *Let $u \in C^2([0, +\infty) \times \mathbb{R}^n)$ be solution of*

$$\begin{cases} u_{tt} - \Delta u + u^5 = 0 \\ u(a, x) = \phi(x) \in C^3 \\ u_t(a, x) = \psi(x) \in C^2 \\ Supp(\varphi) \cup Supp(\psi) \subset B_{\frac{a}{2}}(0) \end{cases} \tag{0.14}$$

we have the estimate:

$$(t^2 - |x|^2)|u(t, x)| \leq C \tag{0.15}$$

where $C := C(\varphi, \psi)$

We notice that the above estimate gives the decay rate of $\frac{1}{t}$ on characteristic cones $\{t - |x| = \tau\}$, $\tau \geq 0$.
For the two space dimensions case, Struwe, in 2010 ([42]), was able to show the existence of a unique smooth solution for (0.7)+(0.2) in the class of radial symmetric smooth functions, in spite of the above initial data classification. We present the result provided by Struwe the following year, in[41], where he was able to avoid the radial condition and to prove the result:

Theorem 0.15. *For any $\varphi, \psi \in C^\infty(\mathbb{R}^2)$ it exists a unique smooth solution $u \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$ for the Cauchy problem (0.7)+(0.2).*

Struwe gives a proposition characterizing blow up phenomenon by energy concentration:

Proposition 0.16. There exists $\epsilon_0 > 0$ such that if a solution $u \in C^\infty([0, T_0) \times \mathbb{R}^2)$ can not be extended to a neighbourhood of $z_0 = (T_0, x_0)$ there holds

$$E(u(t), B_{T_0 - t}(x_0)) \geq \epsilon_0 \qquad \forall t \in [0, T_0) \tag{0.16}$$

It means that whenever a solution of (0.7)+(0.2) blows up at a time t the energy will concentrate at the singular point. The proof of Theorem (0.15) is provided by contradiction. We will assume that the solution can not be extended in a neighbourhood of (T_0, x_0) and the absurd will come from the lack of energy concentration.
The interesting fact comes from the absence of assumptions on the initial data energy. The smart idea comes from a clever extension of the Moser-Trudinger inequality

Lemma 0.17. *For any $E > 0$, $p < +\infty$, there exists $\epsilon = \frac{4\pi^2}{p^2 E}$ and a constant $C > 0$ such that for any $\xi_0 > 0$, any $v \in H_0^1([0,1]^2)$ with*

$$\int_0^1 \int_0^1 (\xi_0 |v_y|^2 + \xi_0^{-1} |v_x|^2) dx dy \leq E, \qquad \int_0^1 \int_0^1 \xi_0^{-1} |v_x|^2 dx dy \leq \epsilon$$

there holds $\int_0^1 \int_0^1 e^{pv^2} dxdy \leq C$

needed to prove the lemma

Lemma 0.18. *There exists* $\epsilon > 0$ *and* $0 < C < +\infty$ *such that for* $T_\epsilon \leq T < T_0$ *there holds*

$$\int_{K_T(z_0)} e^{4u^2} dxdt \leq C(T_0 - T).$$

that improve the bound on the non linear term of the equation (0.7). This result, together with a decay lemma applied to estimate the energy term coming from the nonlinearity, let us reach a uniform smallness condition on the energy of solution inside backward light cone with vertex in (T_0, x_0) for times close to T_0, and rule out the blow up.

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